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Simultaneous Guaranteed-Cost Vector-Optimal Performance Design for Collections of Systems

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The problem of designing a fixed state feedback control law that minimizes an upper bound on linear-quadratic performance measures for m distinct plants is reduced to a convex programming problem. Two methods of reducing conservatism are considered. First, distinct guaranteed-cost bounds are assumed for the individual systems. Second, the performance index is optimized as a vector-valued quantity, and provisions are made to deal with the partially ordered objective space that this implies. The contributions of the paper include the reformulation of the nonlinear programming (NLP) problem of reduced conservatism as one which is solvable with (linear) semidefinite programming software. The advantages are that feasible solutions are found more readily and that the problem is optimized more efficiently. Further, solutions are guaranteed to be globally optimal, whereas those found using NLP are guaranteed only to be locally optimal. Systems engineering tradeoffs between different optimized systems are easily computed with a graphical representation of the set of all noninferior (Pareto optimal) solutions to the problem. A trajectory optimization example is given.

I. Introduction

THE problem of the simultaneous stabilization of a set of state-feedback systems with a single constant feedback gain is important in control design. It typically arises in robust control[1] and in fuzzy control.[2,3] In such applications, a set of m different systems, each uniquely identified by the subscript j, for all $j \in I_m \triangleq \{1, \ldots, m\}$, is considered. The set of linearized state-space representations of the set of m systems to be controlled is

$$\dot{\boldsymbol{x}}_j(t) = \boldsymbol{A}_j \boldsymbol{x}_j(t) + \boldsymbol{B}_j \boldsymbol{u}_j(t), \qquad \forall j \in I_m \tag{1}$$

where each state $x_j \in \mathbb{R}^n$ and each control $u_j \in \mathbb{R}^p$. Each system is to be controlled with the associated member of the collection

$$\boldsymbol{u}_j(t) = -\boldsymbol{K}\boldsymbol{x}_j(t), \qquad \forall j \in I_m \tag{2}$$

each of which shares the same gain matrix $K \in \mathbb{R}^{p \times n}$ with the others. The initial goal is to determine a single gain K that will simultaneously stabilize each of the systems (1).

An example of a collection of systems arises from a nonlinear state trajectory description

$$\dot{\boldsymbol{x}}(t) = f[\boldsymbol{x}(t), \boldsymbol{u}(t), t]$$

linearized about, for example, m different operating points.

A further goal is to choose the simultaneous gain K so that it will control each system optimally in the context of linear-quadratic control. That is, a single gain K is to be chosen that causes control laws (2) to minimize an integral-quadratic cost function, typically,

$$J = \mathbb{E}\left\{ \int_0^\infty [\boldsymbol{x}^T(t)\boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{u}^T(t)\boldsymbol{R}\boldsymbol{u}(t)]\, \mathrm{d}t \right\} \tag{3}$$

The expectation operator $\mathbb{E}\{\}$ is taken over zero-mean random initial conditions $\boldsymbol{x}(0)$ [that is, satisfying $\mathbb{E}\{\boldsymbol{x}(0)\} = \boldsymbol{0}$], which are uniformly distributed over the unit n sphere [that is, satisfying $\mathbb{E}\{\boldsymbol{x}(0)\boldsymbol{x}^T(0)\} = I$].

A deterministic measure of Eq. (3) is minimized via the following theorem.

Theorem 1 (Levine and Athans[4]): For all zero mean stochastic initial conditions $\{\boldsymbol{x}_j(0)\}_{j \in I_m}$ uniformly distributed over the unit n sphere, the expected values of the quadratic cost functions are equal to the traces of the corresponding Lyapunov matrices $\{\boldsymbol{P}_j\}_{j \in I_m}$ as

$$J_j \triangleq \mathbb{E}\left\{ \int_0^\infty \left[\boldsymbol{x}_j^T(t)\boldsymbol{Q}\boldsymbol{x}_j(t) + \boldsymbol{u}_j^T(t)\boldsymbol{R}\boldsymbol{u}_j(t)\right] \mathrm{d}t \right\} = \mathrm{tr}(\boldsymbol{P}_j) \qquad \forall j \in I_m \tag{4}$$

where each member of $\{P_j\}_{j \in I_m}$ satisfies the corresponding member of the Lyapunov matrix equation constraint set

$$\boldsymbol{P}_j(\boldsymbol{A}_j - \boldsymbol{B}_j\boldsymbol{K}) + (\boldsymbol{A}_j - \boldsymbol{B}_j\boldsymbol{K})\boldsymbol{P}_j + \boldsymbol{Q} + \boldsymbol{K}^T\boldsymbol{R}\boldsymbol{K} = 0 \qquad \forall j \in I_m \tag{5}$$

Proof. See Levine and Athans.[4] □

Note that optimal solutions $\{\boldsymbol{P}_j^*\}_{j \in I_m}$ are not necessarily strictly positive definite, which is required[5] for the asymptotic stability of systems (1). Implementations for solving constraints (5) must, therefore, include additional constraints to guarantee such results.

Note also that a set of distinct solutions $\{\boldsymbol{K}, \boldsymbol{P}_j\}_{j \in I_m}$ to the m matrix Lyapunov equations (5) is required because each solution in $\{\boldsymbol{P}_j\}_{j \in I_m}$ is unique. That is, in general there exists no single solution $\boldsymbol{P}$ to the collection of m constraints

$$\boldsymbol{P}(\boldsymbol{A}_j - \boldsymbol{B}_j\boldsymbol{K}) + (\boldsymbol{A}_j - \boldsymbol{B}_j\boldsymbol{K})\boldsymbol{P} + \boldsymbol{Q} + \boldsymbol{K}^T\boldsymbol{R}\boldsymbol{K} = 0, \qquad \forall j \in I_m$$

[note the absence of the index j on matrix $\boldsymbol{P}$, compared to that in Eq. (5)].

Paskota et al.[6] minimize (read: reduce) each of the cost functions (4) by minimizing the sum

$$J(\boldsymbol{K}) \triangleq \sum_{j=1}^{m} \mathrm{tr}(\boldsymbol{P}_j) \tag{6}$$

for single-input/single-output systems using nonlinear programming (NLP) software. However, because this optimization problem is not convex in variables $\boldsymbol{K}$ and $\boldsymbol{P}$, only a local minimum is assured.

In this paper, an extension of another design method proposed by Chang and Peng[7] is used. Calling it guaranteed-cost control, they dealt only with a single system ($m = 1$). In this paper, we bound each of the expected values of cost functions (4) with the single bound $\mathrm{tr}(P)$ satisfying

$$\mathrm{tr}(\boldsymbol{P}_j) \le \mathrm{tr}(\boldsymbol{P}), \qquad \forall j \in I_m \tag{7}$$

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where $\boldsymbol{P}$ satisfies all of the matrix inequality constraints

$$\boldsymbol{P}(\boldsymbol{A}_j - \boldsymbol{B}_j\boldsymbol{K}) + (\boldsymbol{A}_j - \boldsymbol{B}_j\boldsymbol{K})\boldsymbol{P} + \boldsymbol{Q} + \boldsymbol{K}^T\boldsymbol{R}\boldsymbol{K} < 0 \qquad \forall j \in I_m \quad (8)$$

A matrix inequality (8) is defined to mean that the left-hand side is strictly negative definite. Furthermore, when the left-hand side is symmetric, the matrix inequality implies that all of the eigenvalues are in the open left-half plane. Note also that solutions $\boldsymbol{P}$ to matrix inequality (8) are guaranteed to be symmetric and positive definite.

The result is the following optimization problem, which has an objective function linear in $\boldsymbol{P}$.

Optimization Problem 1:

$$\min_{K,P} \operatorname{tr}(\boldsymbol{P}) \quad (9)$$

where $\boldsymbol{P} = \boldsymbol{P}^T > 0$ satisfies constraint set (8).

We call this problem conservative because a single feedback gain $\boldsymbol{K}$ and a single Lyapunov guaranteed-cost bound $\operatorname{tr}(\boldsymbol{P})$ satisfying Eq. (7) are to be found. Furthermore, $\boldsymbol{P}$ must satisfy inequality constraints (8), which are looser than the equality constraints (5).

Minimization of a convex programming problem is both easier and faster than solving one that is not convex. Solutions are also guaranteed to be globally optimal. This paper exploits a matrix change of variables proposed by Bernussou et al.[8] to recast the problem in a convex form solvable by semidefinite programming (SDP) software.[9] Following that, a parameterization of the guaranteed-cost bounds is proposed so that each system can be uniquely bounded from above as

$$J_j = \mathbb{E}\left\{ \int_0^\infty \left[\boldsymbol{x}_j^T(t)\boldsymbol{Q}\boldsymbol{x}_j(t) + \boldsymbol{u}_j^T(t)\boldsymbol{R}\boldsymbol{u}_j(t)\right] \mathrm{d}t \right\} \leq \operatorname{tr}(\boldsymbol{P}_j) \qquad \forall j \in I_m \quad (10)$$

while still ensuring problem convexity.

Finally, consider the following definition.

Definition 1: A vector $\boldsymbol{a}_1 \in \mathbb{R}^n$ is said to be noninferior to vector $\boldsymbol{a}_2 \in \mathbb{R}^n$ if at least one component of $\boldsymbol{a}_2$ is greater than or equal to the corresponding element of $\boldsymbol{a}_1$.

The paper parameterizes and graphically represents the set of all noninferior (in the sense of Pareto[10] and Zadeh[11]), i.e., vector-optimal, values of a vector-valued objective function implied by the set of upper bounds (10)

$$\boldsymbol{J} \triangleq [\operatorname{tr}(\boldsymbol{P}_1) \quad \cdots \quad \operatorname{tr}(\boldsymbol{P}_m)] \in \mathbb{R}^m$$

II. Convex Reformulation

Consider again the conservative set of Lyapunov matrix inequality constraints (8) and introduce matrix variables $\boldsymbol{X} \in \mathbb{R}^{p\times n}$ and $\boldsymbol{Y} = \boldsymbol{Y}^T > 0 \in \mathbb{R}^{n\times n}$. With the rational matrix description change of variables

$$\boldsymbol{P} = \boldsymbol{Y}^{-1}, \qquad \boldsymbol{K} = \boldsymbol{X}\boldsymbol{Y}^{-1} \quad (11)$$

found in Ref. 8, nonlinear inequalities (8) become quadratic in $\boldsymbol{K}$ and $\boldsymbol{P}$ and the basic linear matrix inequality (LMI) lemma, e.g., see Boyd et al.,[1] allows the conversion into equivalent LMIs

$$\begin{bmatrix} -\boldsymbol{A}_j\boldsymbol{Y} + \boldsymbol{B}_j\boldsymbol{X} + \boldsymbol{X}^T\boldsymbol{B}_j - \boldsymbol{Y}\boldsymbol{A}_j^T & \boldsymbol{Y}^T\boldsymbol{Q}^{\frac{1}{2}} & \boldsymbol{X}^T\boldsymbol{R}^{\frac{1}{2}} \\ \boldsymbol{Q}^{\frac{1}{2}}\boldsymbol{Y} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{R}^{\frac{1}{2}}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} > 0 \qquad \forall j \in I_m \quad (12)$$

Linear constraints define a convex search domain. The cost function, on the other hand, becomes

$$\min_{K,P} \operatorname{tr}(\boldsymbol{P}) = \min_{X,Y} \operatorname{tr}(\boldsymbol{Y}^{-1}) \quad (13)$$

which is nonlinear. This objective function, subject to the LMI constraints is a convex programming problem because it can be shown that $\operatorname{tr}(\boldsymbol{Y}^{-1})$ is convex in $\boldsymbol{Y}$. Unfortunately, the available software allows only linear objective functions. In fact, Nesterov and Nemirovskii[12] state that "... to solve a convex problem by an interior point method [as found by Gahinet[9]], we should first reduce the problem to one of minimizing a linear objective over [a] convex domain (which is quite straightforward)."

To deal with this limitation, we instead minimize the trace of a new matrix variable $\boldsymbol{Z} = \boldsymbol{Z}^T > 0$ subject to the nonlinear matrix inequality constraint $\boldsymbol{Z} > \boldsymbol{Y}^{-1}$, which can be rephrased as an LMI as

$$\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{Y} \end{bmatrix} > 0 \quad (14)$$

The resulting numerical SDP optimization problem is summarized as follows.

Optimization Problem 2:

$$\min_{X,Y,Z} \operatorname{tr}(\boldsymbol{Z}) \quad (15)$$

subject to Lyapunov constraints (12), to the slack matrix constraint (14), and to $\boldsymbol{Y} > 0$. Note that $\boldsymbol{Z} > 0$ is implied by Eq. (14), as one of the leading principal minors.

Example 1: An example found in Ref. 6 and a number of other relevant references[13–15] is now used for demonstration. A static state feedback gain matrix $\boldsymbol{K}$ is sought that simultaneously stabilizes four different operating points of an airplane trajectory in the vertical plane and that minimizes the guaranteed-cost bound $\operatorname{tr}(\boldsymbol{P})$. The four operating points are given by a set of four linearized state differential equations (1) assuming a scalar input u. The state coefficient matrices $\{\boldsymbol{A}_j\}_{j\in I_4}$ are

$$\boldsymbol{A}_1 = \begin{bmatrix} -0.9896 & 17.41 & 96.15 \\ 0.2648 & -0.8512 & -11.89 \\ 0 & 0 & -30 \end{bmatrix}$$

$$\boldsymbol{A}_2 = \begin{bmatrix} -0.6607 & 18.11 & 84.34 \\ 0.08201 & -0.6587 & -10.81 \\ 0 & 0 & -30 \end{bmatrix}$$

$$\boldsymbol{A}_3 = \begin{bmatrix} -1.702 & 50.72 & 263.5 \\ 0.2201 & -1.418 & -31.99 \\ 0 & 0 & -30 \end{bmatrix}$$

$$\boldsymbol{A}_4 = \begin{bmatrix} -0.5162 & 26.96 & 178.9 \\ -0.6896 & -1.225 & -30.38 \\ 0 & 0 & -30 \end{bmatrix}$$

and the control coefficient vectors $\{\boldsymbol{b}_j\}_{j\in I_4}$ are

$$\boldsymbol{b}_1 = \begin{bmatrix} -97.78 \\ 0 \\ 30 \end{bmatrix}, \qquad \boldsymbol{b}_2 = \begin{bmatrix} -272.2 \\ 0 \\ 30 \end{bmatrix}$$

$$\boldsymbol{b}_3 = \begin{bmatrix} -85.09 \\ 0 \\ 30 \end{bmatrix}, \qquad \boldsymbol{b}_4 = \begin{bmatrix} -175.6 \\ 0 \\ 30 \end{bmatrix}$$

and weighting matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ are identity matrices. Using the LMI Control Toolbox,[9] optimal points

$$\boldsymbol{X}^* = [-0.2593 \quad 0.0061 \quad 0.0560]$$

$$\boldsymbol{Y}^* = \begin{bmatrix} 3.3514 & -0.3781 & -0.1683 \\ -0.3781 & 0.0569 & 0.0208 \\ -0.1685 & 0.0208 & 0.0387 \end{bmatrix}$$

$$\boldsymbol{Z}^* = \begin{bmatrix} 1.2380 & 7.7939 & 1.2020 \\ 7.7939 & 70.9536 & -4.1979 \\ 1.2020 & -4.1979 & 33.3622 \end{bmatrix}$$

are obtained, resulting in an optimized objective function $\mathrm{tr}(Z^*) = 105.5538$. The optimal Lyapunov matrix is

$$P^* = (Y^*)^{-1} = \begin{bmatrix} 1.2380 & 7.7939 & 1.2020 \\ 7.7939 & 70.9536 & -4.1979 \\ 1.2020 & -4.1979 & 33.3622 \end{bmatrix}$$

leading to $\mathrm{tr}\{P^*\} = 105.5538$. Because $K^* = X^*(Y^*)^{-1}$, the optimal gain is

$$K^* = [-0.2063 \quad -1.8247 \quad 1.5305]$$

III. Simultaneous Control with Reduced Conservatism

The conservative method of solution involves finding a single Lyapunov matrix cost function (9) upper bound through the solution of Optimization Problem 2. Now consider assigning distinct Lyapunov matrix bounds (10), as shown in the following optimization problem. Note that this necessarily implies a vector-valued objective function.

Optimization Problem 3:

$$\min_{K,\{P_j\}_{j\in I_m}} J \triangleq [\mathrm{tr}(P_1) \quad \mathrm{tr}(P_2) \quad \cdots \quad \mathrm{tr}(P_m)]^T \tag{16}$$

where each component $P_j = P_j^T > 0$ satisfies the corresponding member of the collection of Lyapunov constraints

$$P_j(A_j - B_jK) + (A_j - B_jK)^T P_j + Q + K^T RK < 0 \qquad \forall j \in I_m \tag{17}$$

The vector-valued objective function is addressed in the next section. Linearization of the Lyapunov constraints is discussed now.

It has already been shown that LMI constraints can be obtained when

$$P_{j_1} = P_{j_2}, \qquad \forall j_1, j_2 \in I_m \tag{18}$$

The intent now is to reduce conservatism by finding a way to allow $P_{j_1} \neq P_{j_2}$, for at least some distinct system indices j_1 and j_2 in I_m while retaining linearity of constraints (12) and, hence, the convexity of the search domain. The difficulty lies in the requirement for a single simultaneously controlling gain K satisfying Eq. (11) as

$$K = X_1Y_1^{-1} = \cdots = X_mY_m^{-1}$$

It turns out that this causes the linearity of constraints (12) to be lost. The intuitive method of simply enforcing the additional constraints

$$X_1Y_1^{-1} = X_2Y_2^{-1} = \cdots = X_mY_m^{-1}$$

will not suffice because these constraints are not linear in $\{X_j\}_{j\in I_m}$ and $\{Y_j\}_{j\in I_m}$. Another intuitive solution might be to rearrange the constraints as

$$X_1 = KY_1, X_2 = KY_2, \ldots, X_m = KY_m$$

But since K is an optimization variable, these are bilinear.

To recover linear Lyapunov constraints, a parameterization of matrices $\{P_j\}_{j\in I_m}$ is introduced. Let

$$P_j \triangleq \sigma_j P, \qquad \forall j \in I_m \tag{19}$$

where scalar variables $\{\sigma_j\}_{j\in I_m}$ are real and strictly positive. Their substitution into Eq. (17) yields

$$\sigma_j P(A_j - B_jK) + \sigma_j(A_j - B_jK)^T P + Q + K^T RK < 0 \qquad \forall j \in I_m$$

Now replace gain K and Lyapunov matrix P as before with rational matrix parameterizations (11) to obtain

$$\sigma_j\left(A_j^T Y^{-1} - Y^{-1}X^T B_j^T Y^{-1} + Y^{-1}A_j - Y^{-1}B_jXY^{-1}\right) + Q + Y^{-1}X^T RXY^{-1} < 0, \qquad \forall j \in I_m$$

Because Y is necessarily positive definite (because $P = Y^{-1}$ is assumed to be so), congruence transformations can be applied by premultiplying and postmultiplying each term by Y. (This is known to preserve the direction of matrix inequality.) Then each term is multiplied by negative scalar $-\sigma_j^{-1}$ to reverse the direction of inequality to obtain

$$-YA_j^T - A_jY + X^T B_j^T + B_jX - Y\left(\sigma_j^{-1}Q\right)Y - X^T\left(\sigma_j^{-1}R\right)X > 0 \tag{20}$$

and the LMI lemma can be extended to produce the equivalent LMI constraints

$$\begin{bmatrix} -YA_j^T - A_jY + X^T B_j^T + B_jX & YQ^{\frac{1}{2}} & X^T R^{\frac{1}{2}} \\ Q^{\frac{1}{2}}Y & \sigma_j I & 0 \\ R^{\frac{1}{2}}X & 0 & \sigma_j I \end{bmatrix} > 0 \qquad \forall j \in I_m \tag{21}$$

IV. Vector Optimization

Attention now shifts to the optimization of objective function (16). Referred to in the literature as multiple objective or multimodel optimization its meaning, and the partially ordered range spaces that it implies, are now summarized. There is an analogy between the noninferior set and the set of locally optimal solutions to a scalar-valued optimization problem, and it may aid the reader to keep it in mind. The definition of a noninferior point depends directly on the definition of a partially ordered set and how it differs from a completely ordered set. The generalization from an order to a preference order is also reviewed.

A. Theory

We characterize the set of all vector-optimal objective function evaluations in this paper as the intersection of 1) the set of all objective function evaluations that correspond to stable systems (1), that is, they result from all stabilizing gains K, and 2) the set of all objective function evaluations arising from both stabilizing and nonstabilizing gains K, which are not inferior to any of the other objective function evaluations.

We cite a theorem depending on the availability of a globally optimal solution to a scalarization of the vector-valued objective function; it provides a parameterization of the noninferior range space. A convex problem admits such a globally optimal solution and because the multimodel guaranteed-cost problem has been reformulated with linear constraints and objective function means that the requirements of the theorem have been met.

To begin, a partially ordered set is one in which not all members can be said to be 1) preferred to, or better than, or less than, etc., and/or 2) equal to, or equivalent to, etc., the other members of the set. An example of a completely ordered space is the real number line. An example of a partially ordered space is a set of vectors with real and positive components. One vector $a_1 \in \mathbb{R}_+^n$ can be defined to be less than another vector $a_2 \in \mathbb{R}_+^n$ if each component is strictly less than the corresponding element in a_2. But what if some of the components of a_1 are strictly less than their corresponding components in a_2, but the rest are greater than or equal to their counterparts? No decision can be made about whether a_1 is preferred to a_2 or a_2 is preferred to a_1. (Alternatively, definitions based on vector norms do not resolve the dilemma.) In this case we formally state that a_1 is not inferior to a_2 and a_2 is not inferior to a_1. In this sense, a vector optimization problem is solved whenever the entire set of noninferior vector-valued solutions is found, can be parameterized, or can be graphically represented.

The discussion that follows in this section defines the idea of an order on a set with greater rigor and then the idea of a noninferior subset of a partially ordered set. Rudin[16] provides a definition of an order of a set.

Definition 2 (Rudin[16]): Consider a set $\mathcal{J}$. An order on $\mathcal{J}$ is a relation, denoted by $<$, with the following two properties:

1) If $J_1 \in \mathcal{J}$ and $J_2 \in \mathcal{J}$, then one and only one of the statements $J_1 < J_2$, $J_1 = J_2$, $J_2 < J_1$ is true.

2) If $J_1, J_2, J_3 \in \mathcal{J}$, if $J_1 < J_2$ and $J_2 < J_3$, then $J_1 < J_3$.

Although adequate for a set of scalar-valued members, it must be extended for a set of vector-valued members. Sawaragi et al.[17] generalize the idea to that of a preference order. They state that "... the preference order is usually assumed to be at least a strict partial order; that is, irreflexivity of preference (J_1 is not preferred to itself [this is written $J_1 \not\succ J_1$]) and transitivity of preference (if J_1 is preferred to J_2 [written $J_1 \succ J_2$] and if $J_2 \succ J_3$, then $J_1 \succ J_3$) will be supposed." In other words, the notation $J_1 \not\succ J_2$ is used if J_2 is not inferior to J_1. In this paper the efficient set of solutions is the set of all m-tuples

$$\mathcal{J}^* = \left\{\left(\mathrm{tr}\,\boldsymbol{P}_1^*, \mathrm{tr}\,\boldsymbol{P}_2^*, \ldots, \mathrm{tr}\,\boldsymbol{P}_m^*\right) \in \mathbb{R}_+^m\right\}$$

which are not inferior to any other feasible m-tuple. In the following definition, they[17] formally define preference order and the efficient set.

Definition 3 (Sawaragi et al.[17]): Let $\mathcal{J}$ be a feasible set (for example, one which satisfies the Lyapunov LMI constraints (21) and the slack matrix constraint (14) in the objective space $\mathbb{R}^m$, and let $\succ$ be a preference order on $\mathcal{J}$. Then an element $\boldsymbol{J}_1 \in \mathcal{J}$ is said to be an efficient (noninferior) element of $\mathcal{J}$ with respect to the order $\succ$ if there does not exist an element $\boldsymbol{J}_2 \in \mathcal{J}$ such that $\boldsymbol{J}_2 \succ \boldsymbol{J}_1$. The set of all efficient elements is denoted $\mathcal{E}(\mathcal{J}, \succ)$. That is,

$$\mathcal{E}(\mathcal{J}, \succ) = \{\boldsymbol{J}_1 \in \mathcal{J} : \nexists \boldsymbol{J}_2 \in \mathcal{J} \text{ such that } \boldsymbol{J}_2 \succ \boldsymbol{J}_1\} \subset \mathbb{R}^m$$

They[17] further state, "Our aim in a multiobjective optimization problem [is] to find the set of efficient elements (usually not a singleton)."

Next, the space $\mathcal{K}_s$ of stabilizing gains is defined as

$$\mathcal{K}_s \triangleq \{\boldsymbol{K} : \Re[\lambda_i(\boldsymbol{A}_j - \boldsymbol{B}_j\boldsymbol{K})] < 0, \forall i \in I_n, \forall j \in I_m\}$$

where $\Re(\,)$ denotes the real part of a complex number ( ) and $\lambda_i(\,)$ is the ith eigenvalue of a matrix ( ). Given space $\mathcal{K}_s$, we broadly characterize the set of all noninferior objective functions $\boldsymbol{J}$ mapped from set $\mathcal{K}_s$.

Definition 4 (Mäkilä[18]): The range space of the vector-valued objective function $\boldsymbol{J} \in \mathbb{R}_+^m$ over all stabilizing controllers $\boldsymbol{K} \in \mathcal{K}_s$ is defined as

$$\mathcal{R}(\mathcal{K}_s, \boldsymbol{J}) \triangleq \{\boldsymbol{J}(\boldsymbol{K}) : \boldsymbol{K} \in \mathcal{K}_s\} \subset \mathbb{R}^m$$

The set of all efficient gains $\boldsymbol{K}$ is defined as

$$\mathcal{E}(\mathcal{K}, \succ) \triangleq \{\boldsymbol{K}^* \in \mathcal{K}_s : \forall \boldsymbol{K} \in \mathcal{K}_s, \text{ either } \boldsymbol{J}(\boldsymbol{K}^*) \succ \boldsymbol{J}(\boldsymbol{K}), \text{ or } \boldsymbol{J}(\boldsymbol{K}) \not\succ \boldsymbol{J}(\boldsymbol{K}^*)\} \subset \mathbb{R}^{p \times n}$$

The range space of all efficient gains $\boldsymbol{K} \in \mathcal{E}(\mathcal{K}, \succ)$, that is, the space of all efficient m-tuple objective functions $\boldsymbol{J}$ is given as

$$\mathcal{E}(\mathcal{J}, \succ) = \{\boldsymbol{J}(\boldsymbol{K}) : \boldsymbol{K} \in \mathcal{E}(\mathcal{K}, \succ)\} \subset \mathbb{R}^m \tag{22}$$

A constructive characterization of space $\mathcal{E}(\mathcal{J}, \succ)$ is now developed. The next theorem demonstrates that one is possible because of the availability of the convex problem formulation, and Theorem 3, following that, shows how.

Theorem 2 (Mäkilä[18]): Consider a vector m space of multipliers

$$\mathcal{M} \triangleq \left\{\boldsymbol{\mu} \in \mathbb{R}_+^m : \sum_{j=1}^{m} \mu_j = 1\right\} \tag{23}$$

For each $\boldsymbol{\mu} \in \mathcal{M}$, let there exist a [unique] vector

$$\boldsymbol{J}_\mu \in \mathcal{R}(\mathcal{K}_s, \boldsymbol{J}) \subset \mathbb{R}^m$$

satisfying the scalar inequality of inner products

$$\boldsymbol{\mu}^T \boldsymbol{J}_\mu < \boldsymbol{\mu}^T \boldsymbol{J} \in \mathbb{R}, \qquad \forall \boldsymbol{J} \in \{\mathcal{R}(\mathcal{K}_s, \boldsymbol{J}) \backslash \boldsymbol{J}_\mu\}$$

If this condition is satisfied, then the range space of efficient gains is given by

$$\mathcal{E}(\mathcal{J}, \succ) = \{\boldsymbol{J}_\mu : \forall \boldsymbol{\mu} \in \mathcal{M}\}$$

Proof: See Mäkilä.[18] □

Dorato et al.[19] explain that Theorem 2 implies that "if for each $\boldsymbol{\mu} \in \mathcal{M}$, the [scalar] solution [$\boldsymbol{J}_\mu$] is unique," then the range space of the efficient solutions "coincides with a set of scalarized optimization problems. . . ." $\boldsymbol{J}_\mu$ can then be found using the following theorem.

Theorem 3 (Da Cunha and Polak[20]): If scalar function

$$\min_{\boldsymbol{K}, \{\boldsymbol{P}_j\}_{j \in I_m}} \boldsymbol{\mu}^T \boldsymbol{J} = [\mu_1 \quad \mu_2 \quad \cdots \quad \mu_m] \begin{bmatrix} J_1 \\ J_2 \\ \vdots \\ J_m \end{bmatrix}$$

is unique for any fixed vector $\boldsymbol{\mu} \in \mathcal{M}$, then the set of solutions to convex programming problem

$$\min_{\boldsymbol{K}, \{\boldsymbol{P}_j\}_{j \in I_m}} \sum_{j=1}^{m} \boldsymbol{\mu}_j \boldsymbol{J}_j \tag{24}$$

generated by considering all vectors $\boldsymbol{\mu} \in \mathcal{M}$, one at a time, results in the efficient set $\mathcal{E}(\mathcal{J}, \succ)$.

Proof: See Da Cunha and Polak.[20] □

That is, the set of all vectors

$$\left[\boldsymbol{\mu}_1 J_1^* \quad \boldsymbol{\mu}_2 J_2^* \quad \cdots \quad \boldsymbol{\mu}_m J_m^*\right]^T$$

over all

$$\boldsymbol{\mu} = [\boldsymbol{\mu}_1 \quad \boldsymbol{\mu}_2 \quad \cdots \quad \boldsymbol{\mu}_m]^T \in \mathcal{M}$$

in Eq. (23) spans the space of all vector-optimal solutions. The Da Cunha and Polak[20] result is significant because the present work converts a set of nonlinear nonconvex optimization problems for simultaneous system performance design to ones which are convex. Thus, the condition of Theorem 3 requiring that each solution J_μ be unique is automatically satisfied. It is also noted again for emphasis that any optimal point for a convex optimization problem is globally optimal and, in practice, is more easily found than one defined in terms of a nonconvex search space using NLP software.

B. Application

Because of assumed parameterization (19), the m-vector-valued objective function (16) becomes

$$\min_{\boldsymbol{K}, \boldsymbol{P}, \{\sigma_j\}_{j \in I_m}} \boldsymbol{J}' \triangleq [\sigma_1 \,\mathrm{tr}(\boldsymbol{P}) \quad \sigma_2 \,\mathrm{tr}(\boldsymbol{P}) \quad \cdots \quad \sigma_m \,\mathrm{tr}(\boldsymbol{P})]^T \in \mathbb{R}^m \tag{25}$$

Components are bilinear in scalar variables $\{\sigma_j\}_{j \in I_m}$ and matrix variable $\boldsymbol{P}$, but each is strictly positive because all members of $\{\sigma_j\}_{j \in I_m}$ are positive and $\mathrm{tr}(\boldsymbol{P}) > 0$. The available SDP software minimizes only affine functions, and so we choose instead to minimize the $(m+1)$ components of

$$\min_{\boldsymbol{K}, \boldsymbol{P}, \{\sigma_j\}_{j \in I_m}} \boldsymbol{J}'' \triangleq [\sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_m \quad \mathrm{tr}(\boldsymbol{P})]^T \in \mathbb{R}_+^{m+1} \tag{26}$$

to substantially reduce components of $\boldsymbol{J}'$. Whereas this means that conservatism is still present in the problem, not unlike that of Paskota et al.,[6] we are no longer solving an NLP problem but an SDP problem (read: linear programming problem) which 1) produces globally optimal solutions and 2) numerically proceeds more efficiently and reliably.

Next, the space of all vector-optimal solutions to Eq. (26) to Optimization Problem 3 is parameterized. The $\mathcal{M}$ space can be gridded with equidistantly spaced evaluation points. For example, if a collection of $m = 2$ systems were being considered, then each of the values in Table 1 would be used in Eq. (24). The problem becomes more complex, however, as the number of systems m increases. See Table 2 for the case of $m = 3$. Note that $(n+1) + n + (n-1) + \cdots + 2 + 1$ optimization problems need to be solved as opposed to only $(n+1)$ points for the $m = 2$ case. Figure 1 shows that for $m = 3$, the $\mathcal{M}$ space is a plane in 3 space.

For the flight trajectory example, the parameterized efficient space can be graphically represented using Theorem 3 and the following

Table 1 $\mathcal{M}$ **evaluation grid for $m = 2$ systems**

μ_1	μ_2	$\sum_{j=1}^{m=2} \mu_j$
$0/n$	n/n	1
$1/n$	$(n-1)/n$	1
$2/n$	$(n-2)/n$	1
$\vdots$	$\vdots$	$\vdots$
$(n-1)/n$	$1/n$	1
n/n	$0/n$	1

Table 2 $\mathcal{M}$ **evaluation grid for $m = 3$ systems**

μ_1	μ_2	μ_3	$\sum_{j=1}^{m=3} \mu_j$
$0/n$	$0/n$	n/n	1
	$1/n$	$(n-1)/n$	1
	$2/n$	$(n-2)/n$	1
	$\vdots$	$\vdots$	$\vdots$
	$(n-1)/n$	$1/n$	1
	n/n	$0/n$	1
$1/n$	$0/n$	$(n-1)/n$	1
	$1/n$	$(n-2)/n$	1
	$\vdots$	$\vdots$	$\vdots$
	$(n-1)/n$	$0/n$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(n-1)/n$	$0/n$	$1/n$	1
	$1/n$	$0/n$	1
n/n	$0/n$	$0/n$	1

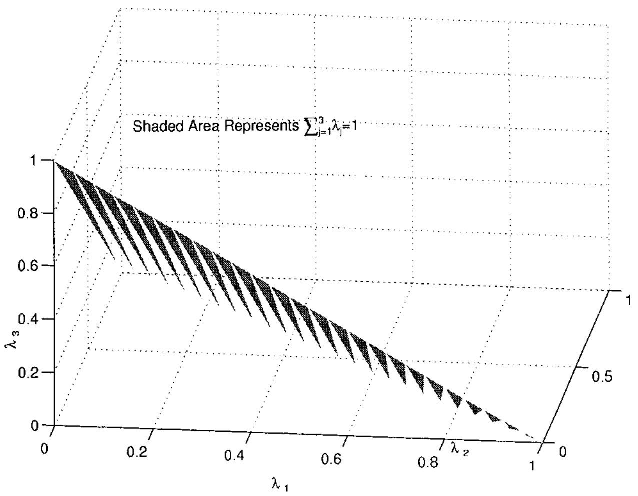

Fig. 1 $\mathcal{M}$ **evaluation space for $m = 3$ systems.**

optimization problem. It is derived from objective function (26) and LMI constraints (21).

Optimization Problem 4:

$$\min_{X,Y,Z,\{\sigma_j\}_{j\in I_m}} \left\{ \left[\sum_{j=1}^{m} \mu_j \sigma_j \right] + \mu_{m+1} \operatorname{tr}\{Z\} \right\} \tag{27}$$

for all $(\mu_1, \mu_2, \ldots, \mu_m + 1) \in \mathcal{M}$, subject to the m Lyapunov LMI constraints

$$\begin{bmatrix} -\boldsymbol{Y}\boldsymbol{A}_j - \boldsymbol{A}_j^T\boldsymbol{Y} + \boldsymbol{X}^T\boldsymbol{B}_j^T + \boldsymbol{B}_j\boldsymbol{X} & \boldsymbol{Y}^T\boldsymbol{Q}^{\frac{1}{2}} & \boldsymbol{X}^T\boldsymbol{R}^{\frac{1}{2}} \\ \boldsymbol{Q}^{\frac{1}{2}}\boldsymbol{Y} & \sigma_j\boldsymbol{I} & 0 \\ \boldsymbol{R}^{\frac{1}{2}}\boldsymbol{X} & 0 & \sigma_j\boldsymbol{I} \end{bmatrix} > 0$$

$$\forall j \in I_m$$

to the single slack LMI constraint

$$\begin{bmatrix} \boldsymbol{Z} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{Y} \end{bmatrix} > 0$$

and to $\boldsymbol{Y} > 0$.

C. Example Revisited

For every gridded vector-valued member of $\mathcal{M}$, a separate and unique minimized objective function (27) is produced. Therefore, the efficient range space is graphically represented with parametric plots consisting of

$$\left[\mu_1\sigma_1^*\right] \quad \text{vs} \quad \left[\mu_2\sigma_2^*\right] \quad \text{vs} \quad \cdots$$
$$\text{vs} \quad \left[\mu_m\sigma_m^*\right] \quad \text{vs} \quad \left[\mu_{m+1}\operatorname{tr}(P^*)\right] \tag{28}$$

From the engineering analysis point of view, plots of

$$\operatorname{tr}\left(\boldsymbol{P}_1^*\right) \quad \text{vs} \quad \operatorname{tr}\left(\boldsymbol{P}_2^*\right) \quad \text{vs} \quad \cdots \quad \text{vs} \quad \operatorname{tr}\left(\boldsymbol{P}_m^*\right)$$

or

$$\sigma_1^*\operatorname{tr}(\boldsymbol{P}^*) \quad \text{vs} \quad \sigma_2^*\operatorname{tr}(\boldsymbol{P}^*) \quad \text{vs} \quad \cdots \quad \text{vs} \quad \sigma_m^*\operatorname{tr}(\boldsymbol{P}^*)$$

reflecting vector-valued objective function (25) would be better. But the requirement for an affine objective function (27) leads us to plot

$$\left[\sigma_1^* + \operatorname{tr}(\boldsymbol{P}^*)\right] \quad \text{vs} \quad \left[\sigma_2^* + \operatorname{tr}(\boldsymbol{P}^*)\right] \quad \text{vs} \quad \cdots \quad \text{vs} \quad \left[\sigma_m^* + \operatorname{tr}(\boldsymbol{P}^*)\right] \tag{29}$$

instead, to represent the noninferior range space.

Next the problem of graphically representing higher dimensional noninferior spaces arises. For instance, the flight trajectory control example discussed earlier involves $m = 4$ systems, so that the plot from Eq. (29) would have to be given in 4 space. It turns out that the sheet represented by Eq. (29) in 4 space can be represented as a volumetric subset of 3 space, but experimentation has led to the conclusion that a series of two-dimensional plots showing for all distinct system indices j_1 and j_2 contained by I_m provide the most intuitively accessible information. For the set of four systems, six such plots are required and are displayed in Figs. 2–7. (Note the different vertical and horizontal scales in Figs. 2–7 as different combinations of systems being simultaneously controlled are shown.) To construct these plots, each multiplier $\boldsymbol{\mu}_j$ interval is gridded into 100 subintervals to achieve apparent smoothness. Again, these plots are graphical representations of the vector-optimal spaces of vector-valued guaranteed-cost bounds of linear-quadratic cost functions (10), subject to the conservatism partially retained by replacing bilinear objective functions (25) with those in Eq. (29). The advantage to the engineer is that system tradeoffs are succinctly represented by these plots.

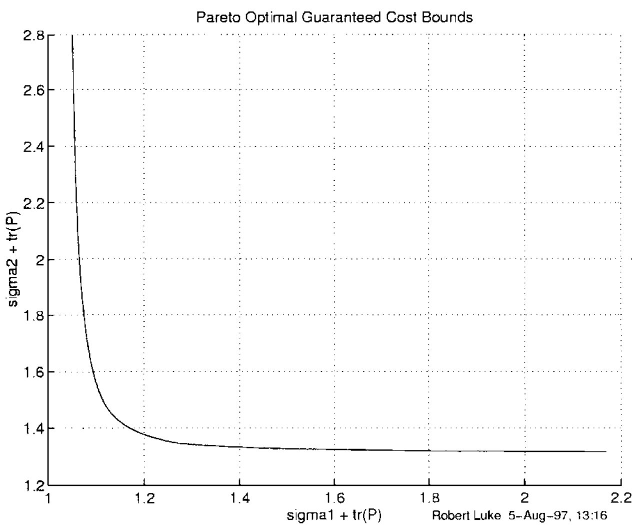

Fig. 2 Flight trajectory example efficient space [$m = 2$, $\sigma_2^* + \operatorname{tr}(P^*)$ vs $\sigma_1^* + \operatorname{tr}(P^*)$].

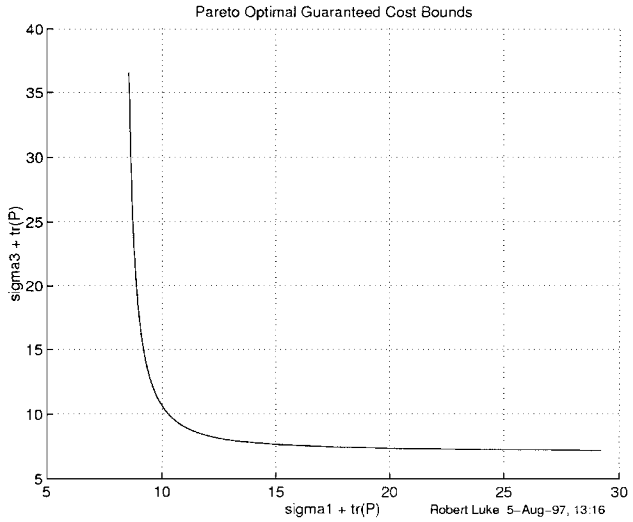

Fig. 3 Flight trajectory example efficient space [m = 2, σ_3^* + tr(P^*) vs σ_1^* + tr(P^*)].

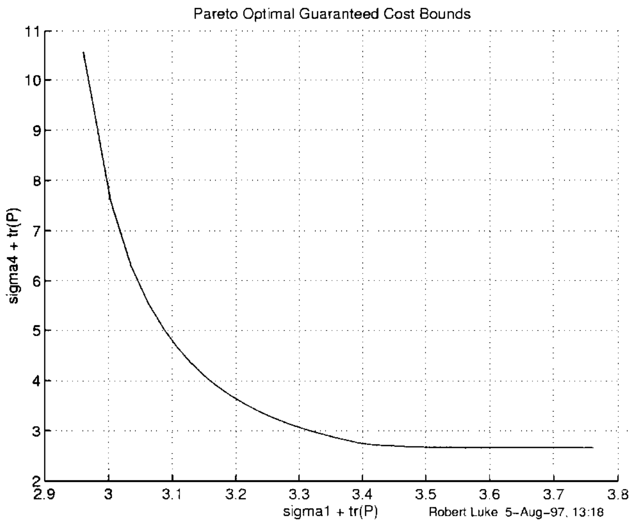

Fig. 4 Flight trajectory example efficient space [m = 2, σ_4^* + tr(P^*) vs σ_1^* + tr(P^*)].

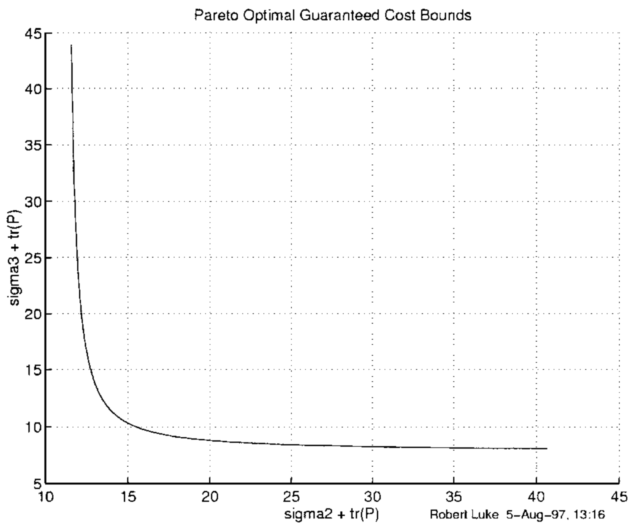

Fig. 5 Flight trajectory example efficient space [m = 2, σ_3^* + tr(P^*) vs σ_2^* + tr(P^*)].

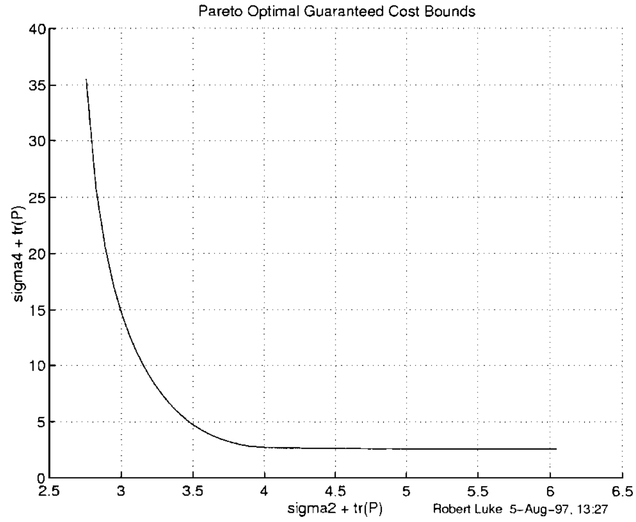

Fig. 6 Flight trajectory example efficient space [m = 2, σ_4^* + tr(P^*) vs σ_2^* + tr(P^*)].

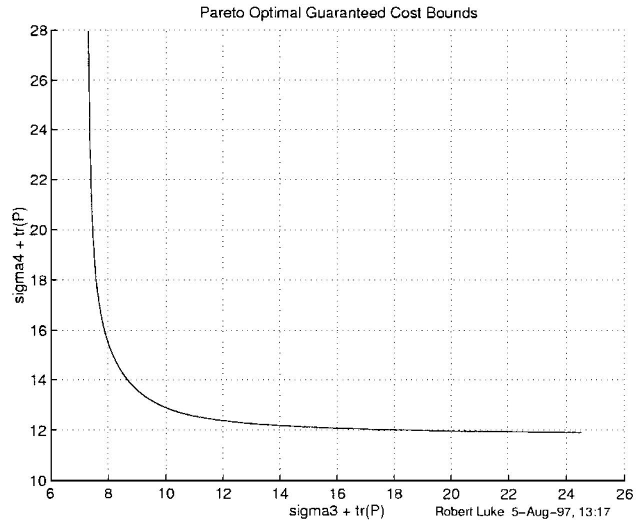

Fig. 7 Flight trajectory example efficient space [m = 2, σ_4^* + tr(P^*) vs σ_3^* + tr(P^*)].

V. Summary

This paper formulates a simultaneous guaranteed-cost control problem for multiple systems as a convex programming problem. The method is demonstrated with a flight trajectory control problem in the vertical plane, which has appeared repeatedly in the literature. An assumed scalar parameterization of the Lyapunov matrix solutions is used to reformulate the NLP problem (with only local minima guaranteed) to a convex SDP problem, which guarantees globally minimal solutions. Then the vector optimization form of the problem, which the reduction in conservatism implies, is analyzed and solved. The result, taking advantage of the unique solutions to convex programming problems, leads to a parameterization of the noninferior range space. This efficient space represents all tradeoffs possible between optimal components of the multiple objective optimization problem. The same example is used to produce such a graphical representation.

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